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Particle–field resonances as multivalued functions of coupling strength and continuum width

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Abstract
For a discrete excited state coupled to a continuum, we prove that the positions of the resonances are, in general, multivalued functions of the coupling constant and the width of the continuum, even if these parameters are real. In such a model, taken from quantum electrodynamics, if the spatial extension of the discrete state is above a certain value, we prove that the bound state which appears has little to do with the excited state. For more general coupling constants, we bring some new information about the behaviour of the resonance corresponding to the excited state when the coupling constant is increased.

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1. Introduction

Many studies are devoted to 'non-perturbative' questions. The present study falls into that category. However, the word 'non-perturbative' may refer to several different things. Non-perturbative properties may be seen as properties which cannot be seen on truncated expansions of various quantities in powers of the coupling constant. For example, the existence of poles of the sum of the series (renormalons, see for instance [1]), or quantities impossible to calculate with such an expansion which vanishes trivially (the energy shift in the double-well problem, see for instance [2]). The word non-perturbative may also be used to refer to the exact determination of the quantities without limitation to any order of their expansions in powers of the coupling constant. This is the case for the exact determination of resonances close to the unperturbed energy level, in atom–radiation interaction with small coupling constant (see [3, 4]). Another example is the non-perturbative approach of radiative decay of [5]. One can also consider as non-perturbative effects which only occur, or are only significant, when the coupling constant is large. The appearance of stable states in a two-level system strongly coupled to a field is an example of such effects (see [6–9], [16, section C_{III}.6]). Our previous studies on the subject [10–15] and the present one are concerned with those kinds of phenomena, although they also yield results in cases where the coupling is small [11]. Indeed, these studies led us to think that the perturbative description, in the sense of a description extrapolated from the description at vanishing coupling, of the states of a quantum system coupled to radiation does not reflect the full structure of the resonances of the global system, which is far richer. This may remind us of the quantum mechanical problem of the double-well, where the degeneracy induced by the two minima complicates the perturbative treatment of the eigenvalue problem (see for instance [2]).

We resume our previous studies on a very simple two-level model, so as to make the description and variation of the resonances in the complex plane more precise, for arbitrary changes of the coupling constant.

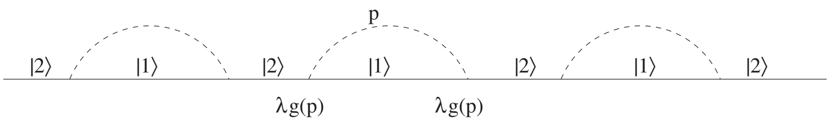
This model of a discrete state coupled to a continuum is presented in [16]. Let us denote this state by $|2\rangle$. Its energy is E, set equal to 1 later on. The system may emit a boson with momentum p and go to the fundamental state denoted by $|1\rangle$; its energy is supposed to be 0. We set $\langle 1, p|H|2\rangle = \lambda g(p)$. Two parameters are important in this problem: the continuum width, i.e. the width of g, and the coupling constant λ. The coupling also depends on the shape of the normed coupling function $g(p)$. The fundamental state energy of the decoupled system remains an eigenvalue of the coupled system Hamiltonian. When the coupling is small, but the smallness depends on g, the first excited state energy E is slightly shifted into the complex lower half-plane. Its position is, to first order in λ,

$$E - 2\lambda^2 \mathcal{P} \int_0^\infty \frac{[g(p)]^2}{p - E}\,\mathrm{d}p - 2\mathrm{i}\pi\lambda^2 [g(E)]^2. \tag{0}$$

When all orders are taken into account, the displaced (complex) energy is the zero near E of the analytic continuation, with respect to z, into the lower half-plane of $f(\lambda, 1, z)$, with

$$f(\lambda, \mu, z) := z - E - 2\lambda^2 \int_0^\infty \frac{[g(p)]^2}{z - \mu p}\,\mathrm{d}p. \tag{1}$$

This expression for $f(\lambda, 1, z)$ comes from the summation of the series in λ for $\langle 2|[z - H]^{-1}|2\rangle = f(z)^{-1}$ (see [16], for instance formulae (6) and (7) of $\mathrm{C_{III}}$). The correction to E is the radiative correction corresponding to the sum of all diagrams of the form



An important point is that $f(\lambda, 1, .)$ (or $f(\lambda, \mu, .)$) has at least another zero, beyond this 'perturbative' one. We started a systematic study of both zeros in previous works.

The main feature of the result of the present paper is that both zeros are to be put on the same footing. They may be transformed into one another by *real* changes in the parameters, λ and the width of the coupling function g, in the same way as the two solutions of a second-order equation depending on a parameter can transform into one another by a analytic change of the parameter. Another way of presenting the result of the paper is to say that the zero commonly affected to the excited state and the zero which is the energy of the stable state mentioned before, studied in strong coupling situations [6, 16], may be identical or not. This indicates a departure from the 'perturbative description'.

We will see that studying this question inevitably leads to consideration of the positions of resonances or bound states as analytic functions of the two above-mentioned parameters.

In previous papers [10, 12], we studied zeros of such functions by following their displacements in the complex plane as a parameter μ varies. This parameter is related to the continuum width (width of g). Focusing on the dependence on this width proved useful

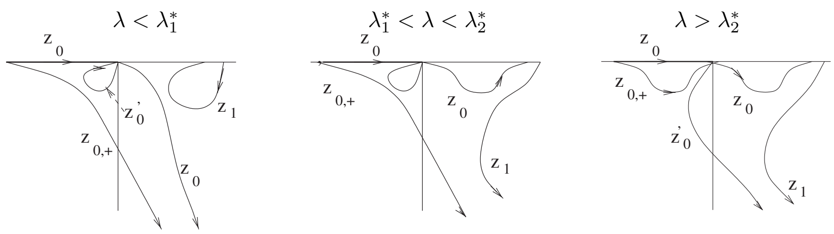


Figure 1. Type of μ-dependence of four zeros of $\hat{f}$, for three different values of λ.

for exhibiting zeros which are not so easily taken into account when the only λ-dependence is considered. In the present paper, we study the positions of the resonances/bound states as functions of these two parameters λ and μ. We show that they are multivalued functions, possibly branches of a unique function, even when the variables are real.

With regard to possible physical applications, we discuss the behaviour of the resonances as the coupling constant increases. In fact, the dependence on λ of the 'perturbative' zero was not a completely solved problem, up to now. In a model where the coupling constant is small, strongly related to QED, we also discuss the behaviour of the resonances as the width of the continuum changes, completing the results of [11]. This change can be performed by varying some physical parameters. The question of finding physical models in which the various types of resonances we exhibit could be studied is now opened.

We start in section 2 with general properties of the zeros giving resonances. Their dependence upon the parameters is then illustrated with numerical examples in section 3. Section 4 is devoted to a discussion of two cases, one in weak coupling, taken from quantum electrodynamics, and the other in strong coupling. The important points for possible applications are stated in propositions 4–6.

2. Resonances and bound states, for a discrete state—mathematical analysis

Let us consider $f(\lambda, \mu, z)$, the function defined by (1) with $E = 1$, for $\mu \neq 0$ and $\mu^{-1}z \notin \mathbb{R}^+ \cup \{0\}$. g has the properties enumerated below. Quantities z, 1 and p are energies, λ and μ are dimensionless parameters. In this paper, we are interested in the zeros of $f(\lambda, \mu, .)$ and its analytic continuations (figure 1). Some of these zeros, among which is the one close to 1 when λ is small, may be obtained by continuity, starting from the zeros for $\mu = 0$. A study of the μ-dependence of the zeros can be found in [10].

That the parameter μ is related to the width of the continuum can be seen by changing the integration variable which changes (1) into

$$f(\lambda, \mu, z) = z - 1 - 2\lambda^2 \int_0^\infty \frac{[g_\mu(p)]^2}{z - p}\, \mathrm{d}p \tag{1'}$$

where $g_\mu(p) := \mu^{-\frac{1}{2}} g(\mu^{-1}p)$ is the function obtained from g by the scaling $p \to \mu p$ of the variable. We thus get all possible values of μ in (1) by considering in (1′) a family of functions g_μ obtained from a unique g by scaling (see also [13]). Function (1′) is precisely the function, a zero of which gives the excited level complex energy when the coupling is λg_μ, g being real. The width of the coupling function is variable. The smaller μ, the smaller the width is.

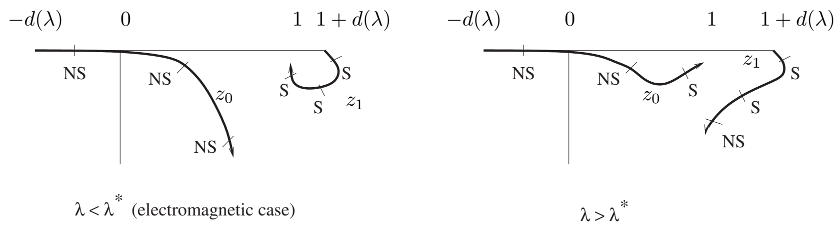


Figure 2. Status of resonances for different values of μ.

We suppose that g has the following properties:

- g is rational with non-real poles π_i.
- $g \in L^2(\mathbb{R}^+)$, $\|g\|_{L^2(\mathbb{R}^+)} = 2^{-1/2}$, $C := 2\int_0^\infty p^{-1}g^2(p)\,\mathrm{d}p < \infty$
- $g(0) = 0$.
- $|pg(p)|$ is bounded for $p \in \mathbb{C}$, $|p| > 2\sup_i |\pi_i|$. This constraint will possibly be removed.

Let us note that these properties are satisfied by the electromagnetic matrix elements given in [17], for example by (8) (formula (3) of [17]), in the case of the hydrogen atom.

For $z = 0$ and $\mu \neq 0$, formula (1) gives $f(\lambda, \mu, 0) = -1 + \mu^{-1}\lambda^2 C$.

We set $\mu_c(\lambda) := C\lambda^2$ and $\lambda_c(\mu) := \{\lambda;\ \mu = C\lambda^2\}$. If $\mu > 0$, we set $\lambda_c^{\pm}(\mu) := \pm C^{-\frac{1}{2}}\mu^{\frac{1}{2}}$. $\mu = \mu_c(\lambda)$, or $\lambda \in \lambda_c(\mu)$ is a necessary and sufficient condition for $f(\lambda, \mu, .)$ to vanish at 0. For example, if $\mu = 1$, one of the resonances will have energy zero and also a vanishing imaginary part for $\lambda = \lambda_c(1)$. It is a transition value of the coupling constant, called critical coupling in [16]. We will meet another singular point $\lambda^*(1)$ later on, and also λ^* (figure 2, see section 4.2 for the meaning of S and NS).

Let $F(\lambda, \mu, .)$ be the multivalued function obtained by analytically continuing $f(\lambda, \mu, .)$. In order to investigate the analytic properties of the zeros of F, we must give the analytic properties of F itself.

Lemma (Analytic properties of $F(\lambda, \mu, z)$, as a function of three complex variables)**.**

(a) Firstly, let (λ, μ) be fixed.
For $\lambda \neq 0$, $\mu \neq 0$, $f(\lambda, \mu, .)$ can be analytically continued in $\mathbb{C}\backslash\{0\}$. It defines a multivalued function having $z = 0$ as a branch point. The only possible other singularities of the various branches are poles at $z = \mu\pi_i$.
$f(0, \mu, z) = z - 1$.
$z \to f(\lambda, 0, z) = z - 1 - \frac{\lambda^2}{z}$ is defined for all $z \neq 0$ and is holomorphic in $\mathbb{C}\backslash\{0\}$ for all λ.

(b) Secondly, let (μ, z) be fixed.
Let us suppose that f or one of its continuations is defined. Then the function is entire in λ. (For $\mu \neq 0$, the function is defined for all z; for $\mu = 0$ it is defined for $z \neq 0$.)

(c) Lastly, let (λ, z) be fixed, with $\lambda \neq 0$.
If $z = 0$, $f(\lambda, ., z)$ is holomorphic in $\mathbb{C}\backslash\{0\}$.
If $z \neq 0$, $f(\lambda, ., z)$ defines a meromorphic multivalued function in μ, certain branches of which have poles $\pi_i^{-1}z$, and having 0 as a branch point.

Proof. Most of the assertions are trivial. To see point (c), let

$$\varphi(\zeta) := \int_0^\infty \frac{g^2(p)}{\zeta - p}\,\mathrm{d}p \tag{2}$$

so that

$$f(\lambda, \mu, z) = z - 1 - 2\frac{\lambda^2}{\mu}\varphi\left(\frac{z}{\mu}\right). \tag{3}$$

Function φ has $\zeta = 0$ as a branch point and poles π_i of g as poles. Therefore, the singular values of $f(\lambda, ., z)$ are $\mu = \pi_i^{-1}z$, on one hand, and $\mu = 0$, which is a branch point, on the other hand. This last point follows from the fact that if $\mu = \epsilon\, \mathrm{e}^{\mathrm{i}\theta}$, $\frac{z}{\mu} = \frac{z}{\epsilon}\,\mathrm{e}^{-\mathrm{i}\theta}$ performs a complete turn around 0 when θ varies from 0 to 2π, which changes the value of $\varphi(z/\mu)$. $\square$

Notation. $\varphi(\lambda, \mu, .)$ and $f(\lambda, \mu, .)$ are obviously defined by (2) and (1) in the plane cut along $\mathbb{R}^+$. The equation $f(\lambda, \mu, z) = 0$ for z is the equation for the eigenvalues of the Hamiltonian coupling the two-level system to the field. It has no solution outside the real numbers, as the Hamiltonian is Hermitian. But when λ increases from 0, not becoming too large (see later on), it is known that the unperturbed eigenvalue 1 is pushed into the lower half-plane, to a point in the second sheet. Let us denote this point by z_2 (the index 2 is chosen here so as to avoid any confusion later on). We get the following behaviour when $t \to \infty$ for the evolution-operator matrix-element

$$\langle e|U(t)|e\rangle = \int_{C_-} \langle e|[z - H]^{-1}|e\rangle\, \mathrm{e}^{-\mathrm{i}tz}\, \mathrm{d}z \sim \mathrm{e}^{-t|\mathrm{Im}\, z_2|}$$

$\mathrm{Im}\, z_2$ being thus related to the life-time of the state $|2\rangle$. Therefore, we will mostly be interested in the continuation which is defined in the plane cut along $\mathbb{R}^-$, from values in the upper half-plane. (The upper part of the cut is included.) We will denote these functions by $\hat{\varphi}(\lambda, \mu, .)$ and $\hat{f}(\lambda, \mu, .)$. Incidently, it is to be kept in mind here that we do not exclude real eigenvalues of the Hamiltonian (see [6, 9]). This will be explained later on. If $\mathrm{Im}\, z < 0$, we have

$$\hat{f}(\lambda, \mu, z) = f(\lambda, \mu, z) + 4\mathrm{i}\pi\frac{\lambda^2}{\mu}g^2\left(\frac{z}{\mu}\right). \tag{3$'$}$$

Let us consider in $(\mathbb{C}\backslash\{0\})^3$ the following system of equations in (λ, μ, z):

$$\begin{cases} z - 1 - 2\frac{\lambda^2}{\mu}\hat{\varphi}\left(\frac{z}{\mu}\right) = 0 \\ \hat{\varphi}'\left(\frac{z}{\mu}\right) - \frac{\mu^2}{2\lambda^2} = 0. \end{cases} \tag{Σ}$$

It defines the singular solutions of (1). (Σ) is equivalent to a system of four real equations for six real unknown variables, the real and imaginary parts of λ, μ and z. We are going to examine three cases in which the number of real unknown variables is reduced to four. In the first one λ is fixed (section 2.1 and numerical example of section 3.5). In the second one, μ is fixed (section 2.2 and examples of section 3.4). In the third one, we require λ and μ to be real (section 2.3, sections 3.2 and 3.3).

We now come to the analyticity properties of the position of the resonances, the zeros of $F(\lambda, \mu, .)$.

2.1. Some singularities $\mu_i^*(\lambda)$ of the zeros, λ being fixed

Definition. *We fix λ and denote by $\{(\mu_i^*(\lambda), z_i^*(\lambda)); i = 1, \ldots, k\}$ the set of solutions for (Σ) in $(\mathbb{C}\backslash\{0\})^2$. We suppose that it is a discrete set containing k elements and set $\mu^*_{i=1,\ldots,k}(\lambda) := \cup_{i=1,\ldots,k}\mu_i^*(\lambda)$.*

Proposition 1 (Analyticity of the zeros of $f(\lambda, \mu, .)$ with respect to μ)**.** *Let λ be fixed in $\in \mathbb{C}\backslash\{0\}$.*

Let $\mu_0 \in \mathbb{C}$ *satisfy* $\mu_0 \neq \mu_i^*(\lambda)$, $\mu_0 \neq 0$ *and* $\mu_0 \neq \mu_c(\lambda)$. *Choose* z_0 *such that* $F(\lambda, \mu_0, z_0) = 0$.

Then we can extend this zero locally, that is to say there exist a neighbourhood $\mathcal{V}_\lambda(\mu_0)$ *and a function* $\mu \to z_{\text{null}}(\lambda, \mu)$ *analytic in* $\mathcal{V}_\lambda(\mu_0)$ *such that* $F(\lambda, \mu, z_{\text{null}}(\lambda, \mu)) = 0$ *for all* μ *in* $\mathcal{V}_\lambda(\mu_0)$ *and* $z_{\text{null}}(\lambda, \mu_0) = z_0$.

Globally, this zero can be analytically continued as a function of μ *along any path in* $\mathbb{C}\backslash\{0, \mu^*_{i=1,\ldots,k}(\lambda), \mu_c(\lambda)\}$ *into a function* $z_{\text{null}}(\lambda, .)$ *satisfying* $F(\lambda, \mu, z_{\text{null}}(\lambda, \mu)) = 0$.

By following the variation with respect to μ of any zero $z_{\text{null}}(\lambda, \mu_0)$, for $\mu_0 \in \mathbb{C}\backslash\{0, \mu^*_{i=1,\ldots,k}(\lambda), \mu_c(\lambda)\}$, this enables us to define a function holomorphic in $\mathbb{C}\backslash\{0, \mu^*_{i=1,\ldots,k}, (\lambda), \mu_c(\lambda)\}$, possibly branching at $\mu_i^*(\lambda)$, $\mu_c(\lambda)$, 0.

Proof. The first part of the proof is a consequence of the fact that z_0 necessarily differs from the poles $\mu_0\pi_i$, and from 0, since $\mu_0 \neq \mu_c(\lambda)$. Then, μ_0 being different from 0, $f(\lambda, \mu, z)$ is holomorphic in μ in a neighbourhood of (μ_0, z_0) from (c). Besides, $f(\lambda, \mu, z)$ is holomorphic in z in a neighbourhood of (μ_0, z_0) from (a). Lastly, since

$$\partial_z f(\lambda, \mu, z) = 1 - \frac{2\lambda^2}{\mu^2}\hat{\varphi}'\left(\frac{z}{\mu}\right) \tag{4}$$

and μ differs from the singular values $\mu_i^*(\lambda)$, then $\partial_z f(\lambda, \mu_0, z_0) \neq 0$ holds. Consequently, the implicit function theorem applies. It yields the local existence of $z_{\text{null}}(\lambda, .)$. Let us now prove the second part. Let us assume that $z_{\text{null}}(\lambda, .)$ is analytic in a disc $D_r(\mu_0)$ around μ_0, with radius r and contained in $\mathbb{C}\backslash\{0, \mu^*_{i=1,\ldots,k}(\lambda), \mu_c(\lambda)\}$. Let γ be a path in $\mathbb{C}\backslash\{0, \mu^*_{i=1,\ldots,k}(\lambda), \mu_c(\lambda)\}$. Choose $\mu' \in D_r(\mu_0) \cap \gamma$. We repeat the preceding argument, replacing (μ_0, z_0) by (μ', z'). Thus, we get a new neighbourhood $\mathcal{V}'(\mu')$ and the function $z_{\text{null}}(\lambda, .)$ is analytically continued in $\mathcal{V}(\mu_0) \cup \mathcal{V}'(\mu')$. The point is to show that the process may be carried out up to any point at a finite distance in $\mathbb{C}\backslash\{0, \mu^*_{i=1,\ldots,k}(\lambda), \mu_c(\lambda)\}$. This is done by copying the argument in [10], replacing the Hurwitz theorem by the implicit function theorem, since the latter now applies for the values of μ we now consider. In short, the reasoning is as follows: should we come up in the step-by-step construction against a limit point which is not the end of the path, this point would nevertheless be regular and it would then still be possible to perform the construction beyond it. □

That some $\mu_i^*(\lambda)$'s are indeed singularities of the functions thus defined will be seen in the numerical examples of section 3.

We now allow the starting point of the μ-variation to be 0 and obtain the essential point in our construction.

Corollary. (Two particular zeros of $F(\lambda, \mu, .)$**).**
Choose $\lambda \neq 0$ *and set* $d(\lambda) := \frac{1}{2}(\sqrt{1 + 4\lambda^2} - 1)$.
Let μ_0 *be a point in* $\mathbb{C}\backslash\{0, \mu^*_{i=1,\ldots,k}(\lambda), \mu_c(\lambda)\}$.
Let γ *be a path in* $\mathbb{C}\backslash\{0, \mu^*_{i=1,\ldots,k}(\lambda), \mu_c(\lambda)\}$ *joining* 0 *to* μ_0. *(Note that this path may turn once or several times around the origin, or around the other singular points* $\mu_i^*(\lambda)$, $\mu_c(\lambda)$.*)*
The two zeros

$$z_0(\lambda, 0) := -d(\lambda) \qquad z_1(\lambda, 0) := 1 + d(\lambda)$$

of $f(\lambda, 0, .)$ *can be continued for* $\mu \in \gamma$ *as two functions of* μ, *holomorphic at any point* $\mu \neq 0$. *Analytic continuations along two homotopic paths give the same result.*

Proof of the corollary. Firstly, we apply the Hurwitz theorem for $\mu \in \gamma \cap \mathcal{V}$, $\mathcal{V}$ being a small neighbourhood of $\mu = 0$. We thus get two zeros for μ on the path near 0, coinciding respectively with $z_0(\lambda, 0)$ and $z_1(\lambda, 0)$ for $\mu = 0$. Let us denote them by $z_0(\lambda, \mu)$ and $z_1(\lambda, \mu)$. (It is not possible to use the implicit function theorem at $\mu' = 0$ because f is not differentiable with respect to μ at this point.) The preceding proposition then allows us to analytically continue these zeros. □

Remark about the notation. We will still denote these zeros by

$$\mu \to z_0(\lambda, \mu) \quad \text{and} \quad \mu \to z_1(\lambda, \mu)$$

for non-small values of μ, but, from now on, in a special case. Without such a precaution, the notation might lead to inconsistencies, due to the multivaluedness of the functions, a point we will be developing in section 3. From this property it indeed follows that $z_0(\lambda, \mu)$, analytically continued around a branch point such as $\mu_i^*(\lambda)$, may change into the principal branch of $z_1(\lambda, \mu)$, or some other zero. The case in which the above notation will be used is, except in section 3.5.1, the following: μ_0 will be real positive, different from $\mu_c(\lambda)$ and, for $\lambda = \lambda_1^*$ (see section 2.3), will be smaller than μ_1^* (section 2.3). The μ-path will be real, unless it encounters $\mu_c(\lambda)$, in which case it will avoid this point by going round along a small half-circle in the upper half-plane.

It is very important to note that here λ is fixed. When the λ-dependence of the zeros of $F(\lambda, \mu, .)$ is discussed, we will introduce a different notation, namely $\xi_{\text{null}}(\lambda, \mu)$, for the zeros.

2.2. Some singularities $\lambda_i^*(\mu)$ of the zeros, μ being fixed

Definition. *Let us denote by $\{(\lambda_i^*(\mu), z_i^*(\mu)); i = 1, \ldots, k'\}$ the set of solutions for (Σ) in $(\mathbb{C}\backslash\{0\})^2$. We again suppose that this set has a finite number of elements, say k'.*

Proposition 2 (Analyticity of the zeros with respect to λ)**.**
Take $\mu \in \mathbb{C}\backslash\{0\}$.
Choose $\lambda_0 \in \mathbb{C}\backslash\{0\}$ such that $\lambda_0 \neq \lambda_i^*(\mu), i = 1, \ldots, k'$ and $\lambda_0 \notin \lambda_c(\mu)$.
Choose z_0 such that $F(\lambda_0, \mu, z_0) = 0$. Then we can extend this zero locally, that is to say there exist a neighbourhood $\mathcal{W}_\mu(\lambda_0)$ and a function $\lambda \to \xi_{\text{null}}(\lambda, \mu)$ analytic in $\mathcal{W}_\mu(\lambda_0)$ such that $F(\lambda, \mu, \xi_{\text{null}}(\lambda, \mu)) = 0$ for all λ in $\mathcal{V}_\mu(\lambda_0)$ and $\xi_{\text{null}}(\lambda_0, \mu) = 0$.
Globally, this zero can be analytically continued as a function of λ, along any path in $\mathbb{C}\backslash(\{0, \lambda^*_{i=1,\ldots,k}(\mu)\} \cup \lambda_c(\mu))$, into a function $\xi_{\text{null}}(., \mu)$ satisfying $F(\lambda, \mu, \xi_{\text{null}}(\lambda, \mu)) = 0$.

It is unavoidable to introduce the new notation $\xi_{\text{null}}(., \mu)$ in order not to let the reader believe that the continuation of $z_0(\lambda, \mu)$ from λ to λ' is $z_0(\lambda', \mu)$. We will see that the continuation may be $z_1(\lambda', \mu)$. Moreover, we find the same difficulty as with the μ-dependence in giving names to the different zeros.

Proof. The first part is proved in the same way as the first part of proposition 1. $f(\lambda, \mu, z)$ is holomorphic in (λ, z) in the neighbourhood of (λ_0, z_0) and $\partial_z f(\lambda_0, \mu, z_0) \neq 0$. Therefore we can define $\xi_{\text{null}}(., \mu)$ in the neighbourhood of 0.

As regards the second part, the line of the proof is the same as that of proposition 2. Let us consider a path in the λ complex plane, starting at λ_0 and ending at a point we call λ_e. Let us assume that γ does not meet any of the points $0, \lambda_i^*(\mu), \lambda^{\pm}(\mu)$. As in the μ-variable case, step by step, we construct an extension $\xi_{\text{null}}(\lambda, \mu)$ around a sequence $\lambda_0, \lambda_1, \ldots$ of points along γ. We have $\xi_{\text{null}}(\lambda_0, \mu) = z_0$ and set $\xi_i := \xi_{\text{null}}(\lambda_i, \mu)$. Let us forget μ in the notation. Suppose that the λ_i accumulate on a point $\lambda_{\text{lim}} \neq \lambda_e$. The hypotheses on g imply that φ, defined by (2), is bounded for sufficiently large ζ (see lemma 1 in [10]) and thus that the ξ_i are bounded. Therefore they have a limit point ξ_{lim}. That ξ_{lim} is not a pole of $F(\lambda_{\text{lim}}, \mu, .)$ results from (3).

From the choice of γ, $\xi_{\text{lim}} \neq 0$. From the continuity of F in (λ, z) wherever F is defined, we get $F(\lambda_{\text{lim}}, \mu, \xi_{\text{lim}}) = 0$. Since $F(., \mu, .)$ is analytic in a neighbourhood of $(\lambda_{\text{lim}}, \xi_{\text{lim}})$ and $(\partial_z F)(\lambda_{\text{lim}}, \mu, \xi_{\text{lim}}) \neq 0$, one can again use the implicit function theorem to define $\xi_{\text{null}}(\lambda)$ beyond λ_{lim}. □

That some $\lambda_i^*(\mu)$ are indeed singularities will be clear with the numerical examples of section 3.

The situation here is somewhat different from that we had when μ was the variable. Indeed, if we start with the value $\lambda = 0$, we can only follow one zero, since there is only one that equals 1. However, when λ equals λ_0 and after we have obtained two zeros by means of the μ-variation, we can use the preceding proposition to follow the zeros when λ varies. This is indeed the essential point in the method used in [10–15]; the method is non-perturbative in the interaction at the very beginning. We thus obtain a function in $\mathbb{C}\backslash(\{0, \lambda^*_{i=1,\dots,k}(\mu)\} \cup \lambda_c(\mu))$ that possibly branches at points in $\{0, \lambda^*_{i=1,\dots,k}\} \cup \lambda_c(\mu)$.

Let

$$S_c := \{(\lambda, \mu) \in (\mathbb{C}\backslash\{0\})^2; \mu = C\lambda^2\}$$

$$\Sigma^* := \{(\lambda, \mu) \in (\mathbb{C}\backslash\{0\}^2); \exists z \text{ s.t. } (\lambda, \mu, z) \text{ is a solution for } (\Sigma)\}.$$

Gathering the results of propositions 1 and 2, we finally get the following method for finding and following zeros of $f(\lambda, \mu, .)$.

By following zeros of $f(\lambda, \mu, .)$, first when μ varies starting from 0, *then when λ varies, we possibly get multivalued functions of two complex variables λ, μ, defined in $(\mathbb{C}\backslash\{0\}^2)\backslash(S_c \cup \Sigma_*)$. This is beyond the scope of the present paper to prove or even discuss theoretically whether these different functions are different branches of a unique multivalued analytic function. The numerical examples in section 3 are a step in that direction.*

The method we proposed in [10] for getting zeros of $f(\lambda, \mu, .)$ is essentially the same as that presented here. It differs only in that it coped with possible multiple zeros, and to do that, we had to lose precision on the results. The possibility of multiple zeros is now excluded by the choice of the paths γ, thanks to the proposition that follows. Moreover, the present study also brings information about the analyticity of the zeros that was not present in [10].

Proposition 3 (Necessary condition for getting a multiple zero)**.** *Let us suppose that we have followed a certain zero $z_{\text{nul}}(\lambda, \mu)$ by the method explained in* [10], *when μ varies up to $\mu_0 > 0$, $\mu_0 \neq \mu_c(\lambda)$. We have thus continued $z_{\text{nul}}(\lambda, \mu)$ in the plane cut along $\mathbb{R}^-$. Let us suppose that $z_{\text{nul}}(\lambda, \mu_0)$ is a multiple zero. Then $(\lambda, \mu_0, z_{\text{null}}(\lambda, \mu_0))$ is a solution for (Σ) and, generically,*

$$\lim_{\mu\to\mu_0} |\partial_\mu z_{\text{nul}}(\lambda, \mu)| = \infty \qquad \lim_{\mu\to\mu_0} |\partial_\lambda z_{\text{nul}}(\lambda, \mu)| = \infty.$$

Proof. If the zero is multiple, then $\partial_z \hat{f}(\lambda, \mu_0, z_{\text{null}}(\lambda, \mu_0)) = 0$, given that if that quantity was not 0, there would be a local isomorphism $\mu \to z_i(\lambda, \mu)$, according to the implicit function theorem. Thus $(\lambda, \mu_0, z_{\text{null}}(\lambda, \mu_0)) \in \Sigma^*$.

Besides, one has

$$\left(\hat{\varphi}'\left(\frac{z_{\text{nul}}(\lambda, \mu)}{\mu}\right) - \frac{\mu^2}{2\lambda^2}\right)\partial_\mu z_{\text{nul}}(\lambda, \mu) = \hat{\varphi}\left(\frac{z_{\text{nul}}(\lambda, \mu)}{\mu}\right) + \frac{z_{\text{nul}}(\lambda, \mu)}{\mu}\hat{\varphi}'\left(\frac{z_{\text{nul}}(\lambda, \mu)}{\mu}\right) \tag{5}$$

$$\left(\hat{\varphi}'\left(\frac{z_{\text{nul}}(\lambda, \mu)}{\mu}\right) - \frac{\mu^2}{2\lambda^2}\right)\partial_\lambda z_{\text{nul}}(\lambda, \mu) = -\frac{2\mu}{\lambda}\hat{\varphi}\left(\frac{z_{\text{nul}}(\lambda, \mu)}{\mu}\right). \tag{6}$$

The right-hand side of (5) equals $(2z_{\text{nul}}(\lambda, \mu) - 1)\mu/(2\lambda^2)$ and the right-hand side of (6) equals $-\frac{\mu^2}{\lambda^3}(z_{\text{nul}}(\lambda, \mu) - 1)$, for solutions of (Σ). The proposition is proved if these quantities do not vanish, which is the case for generic g. □

2.3. 'Absolute' singularities λ_i^* and μ_i^* of the zeros

Definition. *We denote by $(\lambda_i^*, \mu_i^*, z_i^*)$ the set of solutions for (Σ), when λ and μ are required to be real. We again assume that it is a discrete set. We use the term 'absolute' in order to point out that these numbers do not depend on anything else than the shape of g. They are not changed by a dilation of that function. They are pure real numbers.*

Let us suppose that λ and μ, considered in sections 2.1 and 2.2, are real. Then if λ is fixed and different from the λ_i^*, $\mu > 0$ is necessarily different from the $\mu_i^*(\lambda)$. As a consequence, $z_0(\lambda, \mu)$ and $z_1(\lambda, \mu)$ are indeed defined on any 'real' path (inverted commas indicate that the path may have to go round $\mu_c(\lambda)$) followed by the μ-variable. If $\lambda = \lambda_i^*$ for a certain i, the situation is less simple. We will postpone its description until we have given numerical examples in the following section.

Note here that this singular value $\mu_c(\lambda)$ is not to be confused with μ_1^*. $\mu_c(\lambda)$ is a value above which z_0 acquires a non-zero negative imaginary part, when μ alone is varied, λ being fixed. For $\mu < \mu_c(\lambda)$, the eigenvector corresponding to the real negative eigenvalue has the form

$$|2\rangle \otimes \Omega_{\text{bos}} + |1\rangle \otimes h(\lambda, \mu)$$

where Ω_{bos} is the vacuum state and $h(\lambda, \mu)$ is a 1-boson state to be determined. Nothing special occurs for z_0 when μ is varied through μ_1^*.

3. Numerical examples

It seems difficult to solve (Σ). Therefore, it is difficult to describe the analyticity properties with respect to (λ, μ) of the zeros precisely for general g. We will limit ourselves in the rest of this paper to illustrations through examples and numerical calculations. They already give important information on the branch points of these functions, in particular real ones. Numerical values of interest for physical problems can already be obtained.

3.1. The functions g

Let us assume that g is real on the real line; thus $|g(p)|^2 = [g(p)]^2$ and (1) is indeed the function which gives the energy shift of the 'excited' level resulting from the coupling to the field via the coupling function g.

Assuming also that the poles are simple, that is to say that

$$g(p) = \sum_i \left(\frac{a_i}{p - u_i} + \frac{\bar{a}_i}{p - \bar{u}_i} \right)$$

we then get an explicit expression for f_+ by means of the functions we now define. For $\operatorname{Im} u \neq 0$, $\zeta \notin \mathbb{R}^+ \cup \{0\}$ and $\zeta \neq u$,

$$h_{uu}(\zeta) := \int_0^\infty \frac{1}{(p-u)^2} \frac{1}{\zeta - p} \, \mathrm{d}p = \frac{\log(-\zeta) - \log(-u)}{(\zeta - u)^2} + \frac{1}{u(u - \zeta)}$$

and for $u \neq v$, $\operatorname{Im} u \neq 0$, $\operatorname{Im} v \neq 0$, $\zeta \neq u$, $\zeta \neq v$

$$\begin{aligned} h_{uv}(\zeta) &:= \int_0^\infty \frac{1}{(p-u)(p-v)} \frac{1}{\zeta - p} \, \mathrm{d}p \\ &= \frac{\log(-\zeta)}{(\zeta - u)(\zeta - v)} + \frac{1}{u - v} \left(\frac{\log(-v)}{\zeta - v} - \frac{\log(-u)}{\zeta - u} \right). \end{aligned}$$

Some calculations may then be quickly performed on machines if g has only two (non-real) poles. Let us denote the one in the lower half-plane by u, the other one being $\bar{u}$. Let us suppose that $a = \alpha\big(1 - \frac{\mathrm{Re}\, u}{\mathrm{Im}\, u}\mathrm{i}\big)$, $\alpha \in \mathbb{R}^+$, so that $g(0) = 0$ holds. Let us also choose α in order that $\|g\|_2 = 1$. Then, for $\zeta \notin \mathbb{R}^+$,

$$\varphi(\zeta) = a^2 h_{uu} + 2|a|^2 h_{u\bar{u}} + \bar{a}^2 h_{\bar{u}\bar{u}}.$$

In what follows, most of the time we will consider u to be $-\mathrm{i}$ and, consequently, $a = \frac{1}{\sqrt{2\pi}}$ and

$$g(p) = \sqrt{\frac{2}{\pi}}\frac{p}{1 + p^2}. \tag{7}$$

A more realistic shape for g can be obtained from the electromagnetic matrix elements given in [17]; with a dimensionless s, it is

$$g(s) = \frac{4}{\sqrt{\pi}}\frac{s}{(1 + s^2)^2}. \tag{8}$$

The pole is not simple.

3.2. *The case where the poles of g are $\pm\mathrm{i}$; study of the zeros, as functions of real λ and μ (except for $\mu \sim \mu_c(\lambda)$)*

We have

$$\varphi(\zeta) = r_1(\zeta)\log(-\zeta) + r_2(\zeta)$$

where

$$r_1(\zeta) := \frac{1}{2\pi}\left(\frac{1}{(\zeta + \mathrm{i})^2} + \frac{2}{(\zeta + \mathrm{i})(\zeta - \mathrm{i})} + \frac{1}{(\zeta - \mathrm{i})^2}\right)$$

$$r_2(\zeta) := \frac{\mathrm{i}}{4}\left(\frac{1}{(\zeta - \mathrm{i})^2} - \frac{1}{(\zeta + \mathrm{i})^2}\right) + \frac{1}{\zeta + \mathrm{i}}\left(\frac{1}{4} - \frac{\mathrm{i}}{2\pi}\right) + \frac{1}{\zeta - \mathrm{i}}\left(\frac{1}{4} + \frac{\mathrm{i}}{2\pi}\right).$$

These expressions, together with (2), give expressions for $f(\lambda, \mu, z)$. Others can be found in [13, formulae (5)–(7)].

We saw that the resonances were constructed by nailing down the coupling constant and varying μ along some paths starting from 0. These paths have to avoid singularities which are solutions of (Σ). We thus have to solve that system.

But this amounts to solving a transcendental equation of the form $\log(\zeta(\lambda, \mu)) = r(\zeta(\lambda, \mu))$ for $\zeta(\lambda, \mu) = \mu^{-1} z_{\mathrm{null}}(\lambda, \mu)$; here r is a rational function with polynomial coefficients in λ and μ, and $\zeta(\lambda, \mu)$ is a solution of an algebraic equation, with polynomial coefficients in λ and μ. We will limit ourselves to an empirical search for solutions on a computer. Let us first examine the case which seems the most interesting physically, namely the one where λ and μ are both real. However, let us note that it is convenient to give μ a small imaginary part near $\mu_c(\lambda)$.

3.2.1. *Real singular values of the coupling constant and of the continuum width.*

A solution of (Σ) obtained by computer is

$$\lambda_1^* = 0.504\,84 \qquad \mu_1^* = 1.081\,08 \qquad z_1^* = 0.752\,92 - 0.471\,43\mathrm{i}.$$

We cannot say at the moment whether there are others or not. When $u = 1 - \mathrm{i}$, at least one other solution does exist (see section 3.3). We are going to see that these values are particularly important in order to be able to follow the displacements of the zeros when λ and μ vary.

The displacement of the resonances is now described concretely in an example. We will see how they interchange. Variation with μ is considered first. We then show the variation with λ, according to the paths in the (λ, μ)-plane drawn in figure 5. The reader may find it helpful to look directly at figures 3–7.

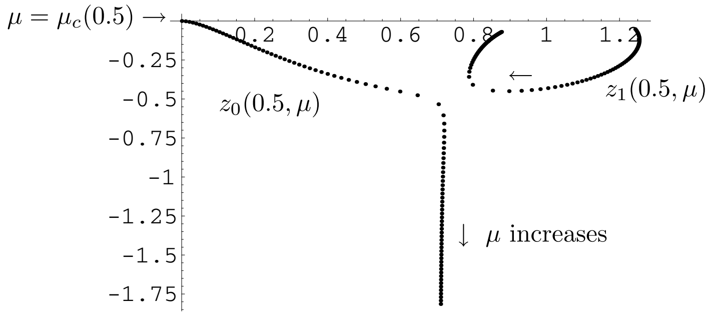


Figure 3. Variation in μ of the resonance positions ((Re(z), Im(z))), for $\lambda = 0.50$.

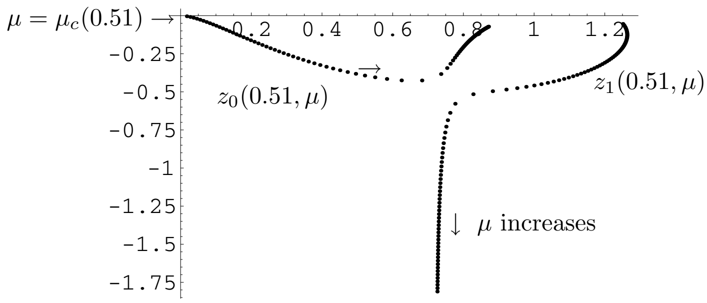


Figure 4. Variation in μ of the resonance positions ((Re(z), Im(z)), for $\lambda = 0.51$.

3.2.2. Construction, notation and analyticity properties of two resonances. Let us first suppose that λ is fixed at a real value not equal to any of the λ_i^*. It will vary in section 3.2.3. We may then use the corollary of proposition 1 and section 2.3 to construct two resonances, for a given positive real value μ_0 of μ, through variation of μ on the reals (except possibly near $\mu_c(\lambda)$). The principle of that construction was sketched in [10] and recalled in the previous pages. The present study brings complements, for example, with regard to the analyticity of their dependence on $\alpha := (\lambda, \mu)$.

Two cases have to be distinguished according to whether μ_0 is greater or smaller than $\mu_c(\lambda)$.

(i) Let us suppose that $\mu_0 > \mu_c(\lambda)$. All paths joining 0 to μ_0 along the reals, except near $\mu_c(\lambda)$ where they follow a small half-circle in the upper half-plane, are homotopic in $\mathbb{C}\backslash\{\mu_i^*(\lambda), \mu_c(\lambda)\}$, provided the radius of the circle is sufficiently small. Homotopic means that they can be transformed continuously into one another. The consequence of the corollary of proposition 1 and of section 2.3 is that the two zeros $z_0(\lambda, 0)$ and $z_1(\lambda, 0)$ may be continued in a unique way as analytic functions $z_0(\lambda, .)$ and $z_1(\lambda, .)$ in some neighbourhood of these paths. At the end of section 2.3, we recalled that $z_0(\lambda, .)$ is a real negative number as long as μ stays smaller than $\mu_c(\lambda)$. It would tend to 0 if μ reached $\mu_c(\lambda)$. As it goes round this point, $z_0(\lambda, \mu)$ goes round 0 clockwise to be in the lower half-plane for $\mu = \mu_c(\lambda) + \epsilon$. $z_1(\lambda, \mu)$ starts from 1 and remains in the lower half-plane. The positions of the two zeros for the values of μ in $[\mu_c(\lambda), 2]$ are given in figures 3 and 4, respectively, for $\lambda = 0.50$ and $\lambda = 0.51$, two values on either side of the critical value λ_1^*. We have $\mu_c(0.50) \simeq 0.159$, $\mu_c(0.51) \simeq 0.166$ and

$$\lim_{\mu\to 0} z_1(\lambda, \mu) = 1 + d(\lambda) \qquad \lim_{\mu\to\mu_c(\lambda)} z_0(\lambda, \mu) = 0.$$

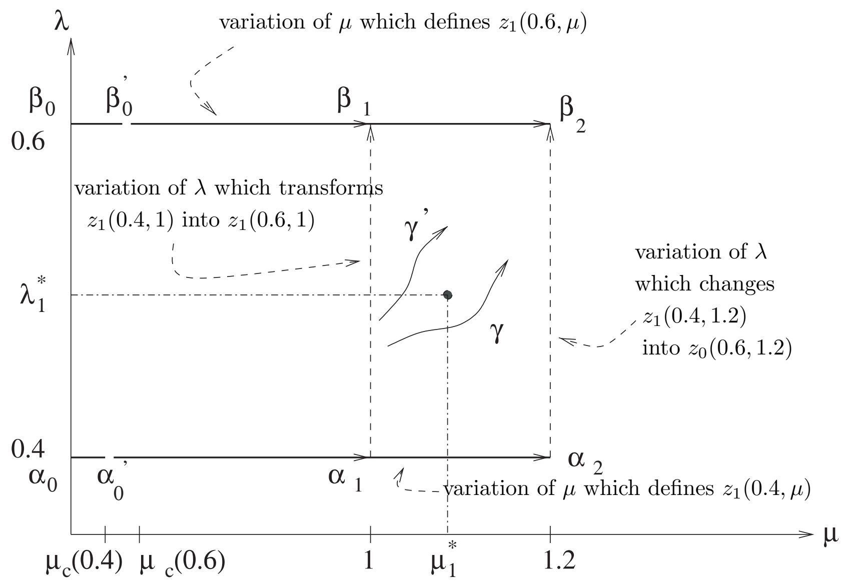


Figure 5. Different paths in the (λ, μ)-plane for constructing resonances.

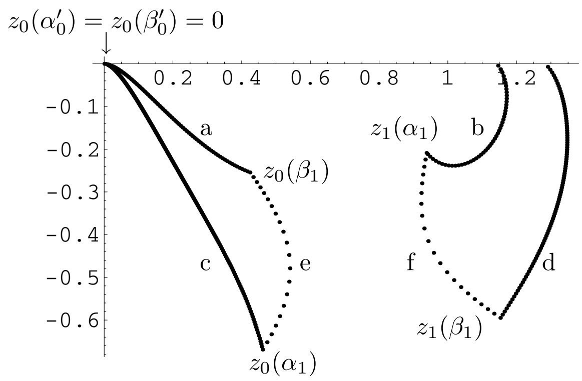


Figure 6. Variations of the resonance position, first with respect to μ up to $\mu_0 = 1 < \mu_1^*$ (bold lines), then with respect to λ (dotted lines).

In these figures it can be seen that the two zeros do not cross. As we said before, the absence of crossing allows the notation.

If $\lambda = \lambda_1^*$, there is a double point for $\mu = \mu_1^*$, and thus an ambiguity in the notation of the zeros for $\mu > \mu_1^*$. In any case, as proved in [10], two zeros still exist for $\mu > \mu_1^*$.

(ii) If $\mu_0 < \mu_c(\lambda)$, the difficulty about $z_0(\lambda, \mu)$ going to 0 for $\mu = \mu_c(\lambda)$ does not occur. $z_0(\lambda, \mu_0)$ is in fact a bound state.

Remarks. Graphs in figures 3 and 4 are obtained for a discrete set of values of μ, and the spreading which can be seen near $0.75 - 0.5\mathrm{i}$ indicates that the derivative with respect to μ

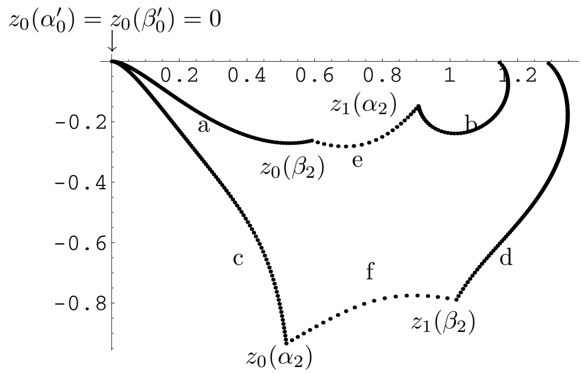


Figure 7. Variations of the resonance position, first with respect to μ up to $\mu_0 = 1.2 > \mu_1^*$ (bold lines), then with respect to λ (dotted lines).

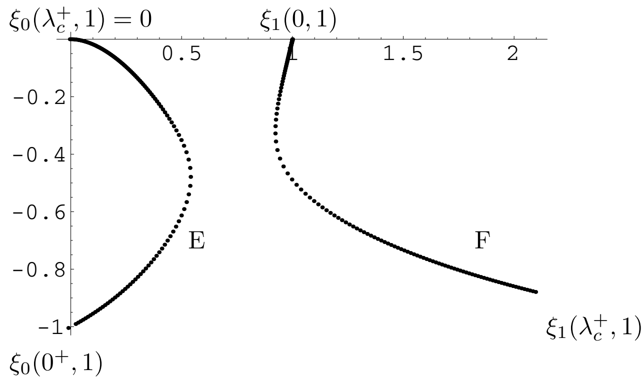


Figure 8. Variations of the resonance positions with respect to λ, for $\mu = 1 < \mu_1^*$.

of the function is getting large. This is in line with proposition 3. Besides, when μ goes to infinity, it can be shown that the zeros enter one of the two cones with arbitrary small aperture containing either $\mathbb{R}^+$ or *u*.

Of these two resonances, one is of course the one we are used to from the perturbative point of view. But this point of view depends on a variation with respect to λ, and we cannot tell which of these resonances we have just obtained corresponds to the excited level until we have discussed the λ-dependence. Let us now examine this question.

3.2.3. Variation with respect to λ (possible resonance interchange). We are going to see a third reason to get interested in arbitrary values, even complex values, of μ. Studying the variation with respect to λ, *after* that with respect to μ, will reveal a multivaluedness and the following related surprising property:

The behaviour of the resonances as λ varies depends on whether μ is smaller or greater than μ_1^*.

We are going to show (in figures 6 and 8) that if μ is fixed at a value lower than μ_1^*, the negative energy stable state which appears when λ increases (see [16]) does not come from the resonance close to 1 when λ is small, but from another one (close to the pole $u = -\mathrm{i}$ of *g*).

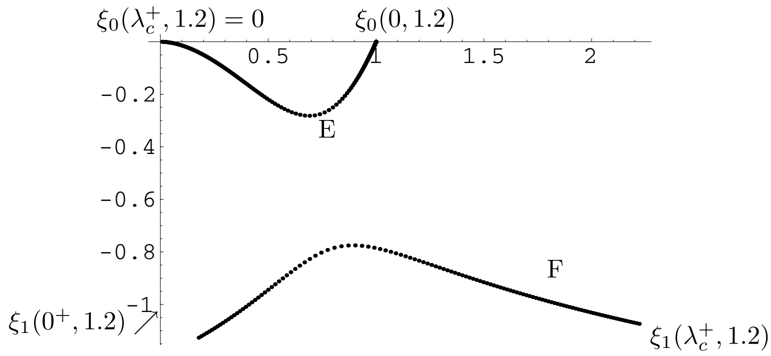


Figure 9. Variations of the resonance positions with respect to λ, for $\mu = 1.2 > \mu_1^*$.

When μ is fixed at a value higher than μ_1^*, it does come from this resonance (close to 1 when λ is small), contrary to what was suggested by the following sentence in [13], section 3.3: 'our study seems to indicate that the zero of f_1 which becomes real negative when λ increases from 0 to a value greater than $\lambda_c(\mu)$ is $z_0(\lambda, \mu)$ rather than $z_1(\lambda, \mu)$'. The statement is correct if $\mu_1 > \mu_1^*$ and false if $\mu > \mu^*$. In [16], it is indeed because the width is supposed to be large (formula (43) of C_{III}.6) that it is the discrete state which has been turned into a stable state when the coupling constant is increased (see end of C_{III}.6).

Let us point out that figure 3 in [13] may be misleading in that it may let one think that the position of the resonances z_0 and z_1 is a (univalued) function of (λ, μ), which is only true if μ is below the critical value μ_1^* (for example if $\mu = 1$).

Let us examine this question in detail in the case $u = -\mathrm{i}$.

Let us fix μ at a value μ_0 close to μ_1^* but different from μ_1^*. Let us consider the variation of the zeros $z_0(\lambda, \mu_0)$ and $z_1(\lambda, \mu_0)$ with respect to λ, from a real starting value λ_0, distinguishing the two cases according to whether μ_0 is smaller or greater than μ_1^*. We will successively take $\mu_0 = 1$ and $\mu_0 = 1.2$, and consider λ varying in an interval containing the singular value λ_1^*. Let us underline the fact that, finally, the zeros have been followed first in μ from 0 to μ_0, λ being fixed at λ_0, then in λ, starting from λ_0. Figure 5 shows the variation paths of the parameter $\alpha := (\lambda, \mu)$ in $\mathbb{R}^2$. The bold printed segments correspond to μ-variations, and those with dashed lines to λ-variations. This format is also used in figures 6 and 7, with dotted lines replacing dashed ones. The discontinuity on the segment $\alpha_0\alpha_1$ (resp. $\beta_0\beta_1$) at α_0' (resp. β_0') indicates that μ must go round the point $\mu_c(0.4)$ (resp. $\mu_c(0.6)$) in the upper half-plane.

Paths followed by the zeros in connection with these variations of λ and μ are shown in figures 6–9. Let us comment on these figures.

(i) $\mu_0 = 1$ (figures 6 and 8).

Notation: $\alpha_1 := (0.4, 1)$ and $\beta_1 := (0.6, 1)$. We have $\lambda_c^+(1) \simeq 1.25$.

- *Obtaining $z_i(\alpha_1)$ and $z_i(\beta_1)$ by varying μ (bold lines in figure* 6). $z_0(\alpha_1)$ and $z_0(\beta_1)$ are obtained through the analytic continuation along a path joining 0 to 1 in the μ-space, a path following the real axis but going round $\mu_c(0.4)$ (resp. $\mu_c(0.6)$). For $\mu = 0$, the zero is $z_0(0.4, 0)$ (resp. $z_0(0.6, 0)$), and at the end of the path, it is $z_0(\alpha_1)$ (resp. $z_0(\beta_1)$). The values for $\mu = \mu_c(0.4) + \epsilon$ and $\mu = \mu_c(0.6) + \epsilon$ are close to 0 in the third quadrant. In fact, we have $z_0(\alpha_0') = z_0(\beta_0') = 0$, where $\alpha_0' := (0.4, \mu_c(0.4))$ and $\beta_0' := (0.6, \mu_c(0.6))$.

In figure 6, the zero $z_0(\alpha)$ follows the segment $]-d(0.4), 0^-[$, goes round 0, then follows arc 'c', whereas the zero $z_0(\beta)$ follows the segment $]-d(0.6), 0^-[$, goes round 0, then follows arc 'a'. We have $d(0.4) \simeq 0.14$ and $d(0.6) \simeq 0.28$. Details of the path followed by the zeros near 0 are not shown.

$z_1(\alpha_1)$ and $z_1(\beta_1)$ are constructed in the same way, but we no longer need to avoid $\mu_c(0.4)$ and $\mu_c(0.6)$ since the zero does not come to the branch point of $f(\lambda, \mu, .)$ for these values of μ. $z_1(\alpha_1)$ is constructed by following arc 'b' whereas $z_1(\beta_1)$ is constructed by following arc 'd'.

– *Variation with respect to* λ (*figures* 6 *and* 8). Starting from $z_0(\alpha_1)$ and $z_1(\alpha_1)$ obtained above, we now vary λ from 0.4 to 0.6 and get arcs 'e' and 'f', respectively, in dotted lines in figure 6. When extending the variation interval of λ to $[0, \lambda_c^+(1)]$, we get curves 'E' and 'F' of figure 8. Arcs 'e' and 'f' are thus respectively parts of 'E' and 'F'. We explained earlier that we have to give new names to the zeros thus obtained by varying λ since there may be interchanges $z_0 \leftrightarrow z_1$; this will occur below in the case $\mu = 1.2$. So let us call them $\xi_0(\lambda, 1)$ ('E' curve) and $\xi_1(\lambda, 1)$ ('F' curve). The important point is the following:

• *When* λ *increases from* 0.4 *to* 0.6, *that is, in figure* 5, *when* α *goes along* $\alpha_1\beta_1$, *then* $z_0(\alpha)$ *continuously varies from* $z_0(\alpha_1)$ *to* $z_0(\beta_1)$. *In the same way*, $z_1(\alpha)$ *continuously varies from* $z_1(\alpha_1)$ to $z_1(\beta_1)$.

The change in the notation is therefore superfluous here.

Incidentally, let us note the following two points. First, on the F curve $\xi_0(\lambda_c^+(1)) = 0$ holds, whereas there is nothing special about the point $\xi_1(\lambda_c^+(1))$. Second,

$$\lim_{\lambda\to 0} \xi_0(\lambda, 1) = -\mathrm{i} \qquad \lim_{\lambda\to 0} \xi_1(\lambda, 1) = 1.$$

(Dotted parts in figures 8 and 9 do not mean anything special.)

(ii) $\mu_0 = 1.2$ (figures 7 and 9)

Notation: $\alpha_2 := (0.4, 1.2)$ and $\beta_2 := (0.6, 1.2)$. We have $\lambda_c^+(1.2) \simeq 1.37$.

– *Obtaining* $z_i(\alpha_2)$ *and* $z_i(\beta_2)$ *by varying* μ (*bold lines in figure* 7)
Same as in the preceding case.

– *Variation with respect to* λ (*figures* 7 *and* 9)
The principle is the same but the result is different:

• *When* λ *increases from* 0.4 *to* 0.6, *that is, in figure* 5, *when* α *goes along the segment* $\alpha_2\beta_2$, *then* $\xi_0(\alpha)$ *continuously varies from* $z_0(\alpha_2)$ *to* $z_1(\beta_2)$. *In the same way,* $\xi_1(\alpha)$ *continuously varies from* $z_1(\alpha_2)$ *to* $z_0(\beta_2)$.

The change in the notation is therefore essential here. For $\lambda \in [0.4, \lambda_1^*[$, $\xi(\lambda, \mu) = z_0(\lambda, \mu)$, whereas for $\lambda \in]\lambda_1^*, 0.6]$, $\xi(\lambda, \mu) = z_1(\lambda, \mu)$. For $\lambda = \lambda_1^*$, z_0 and z_1 are not defined, because of the ambiguity in choosing the arc defining $z_0(\lambda_1^*, \mu)$ at points $\mu > \mu_1^*$. $z_0(\lambda_1^* - 0, \mu)$ and $z_1(\lambda_1^* + 0, \mu)$ are equal.

The preceding result may also be stated in the following terms:

In $(\mathbb{C}\backslash\{0\}^2)\backslash(S_c \cup \Sigma^*)$, let us call γ the path $\alpha_0\alpha_0'\alpha_1\alpha_2\beta_2$ where it is understood that γ goes round α_0', as has been indicated before, and γ' the path $\alpha_0\beta_0\beta_0'\beta_1\beta_2$ with the same precaution at point β_0'. The difference between the two paths γ and γ' is shown graphically in figure 5. They are not homotopic in $(\mathbb{C}\backslash\{0\}^2)\backslash(S_c \cup \Sigma_*)$.

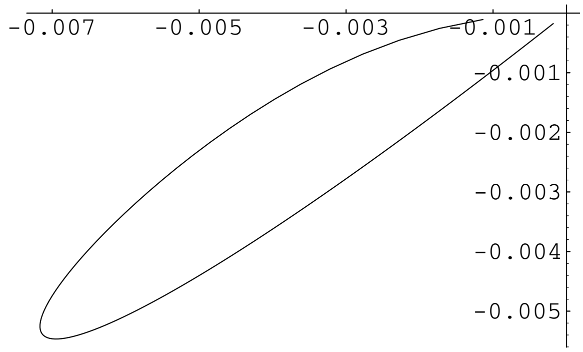


Figure 10. Values of $(\mathrm{Re}\, z_0'(0.4, \mu), \mathrm{Im}\, z_0'(0.4, \mu))$ for $\mu \in [\epsilon, \mu_c(0.4) - \epsilon]$.

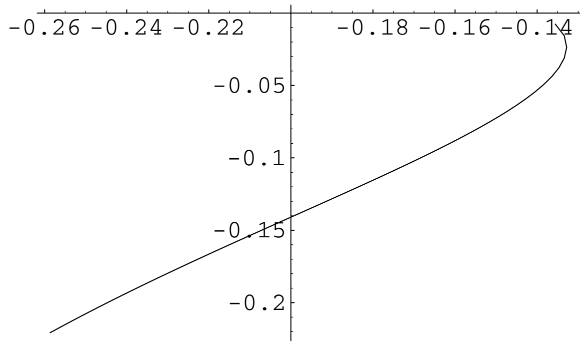


Figure 11. Values of $(\mathrm{Re}\, z_{0,+}(0.4, \mu), \mathrm{Im}\, z_{0,+}(0.4, \mu))$ for $\mu \in [\epsilon, \mu_c(0.4) - \epsilon]$.

Proposition 4 (Multivaluedness of the resonance dependence on the parameters λ, μ)**.** *The two paths γ and γ' define two different analytic continuations for each of $z_0(0.4, 0^+)$ and $z_1(0.4, 0^+)$. For $z_0(0.4, 0^+)$, the values obtained are respectively $z_1(0.6, 1.2)$ for γ (path $[-d(0.4), 0[$* 'c f' *in figure 7) and $z_0(0.6, 1.2)$ for γ' (path $[-d(0.6), 0[$* 'a'*). For $z_1(0.4, 0^+)$, they are $z_0(0.6, 1.2)$, for γ (path* 'b e' *in figure 7) and $z_1(0.6, 1.2)$ for γ' (path* 'd'*).*

Thus $\alpha_1^* := (\lambda_1^*, \mu_1^*)$ appears as a branch point for the multivalued function that the two zeros z_0 and z_1 of $F(\alpha, .)$ define. The order of the branching is 2.

That the two zeros z_0 and z_1 may interchange has the important consequence that we must be very cautious in giving each of the resonances a name. This is particularly true when we want to compare the resonances to what is known from the perturbative approach, which only takes into account variation with respect to the λ parameter. This point will be discussed in section 4.2.

In figures 10 and 11, we present two other zeros of $\hat{f}$. They could be obtained from the preceding ones through analytic continuations. In fact, the function F with an infinite number of branches has an infinite number of zeros but all those considered in the paper are in one unique sheet.

Figure 10 describes a zero that was denoted by $z_0'(\lambda, \mu)$ in [13]. Here $\lambda = 0.4$ and μ varies in $]0, \mu_c(0.4)]$ with

$$\lim_{\mu \to 0} z_0'(0.4, \mu) = 0 \qquad \text{and} \qquad \lim_{\mu \to \mu_c(0.4)} z_0'(0.4, \mu) = 0.$$

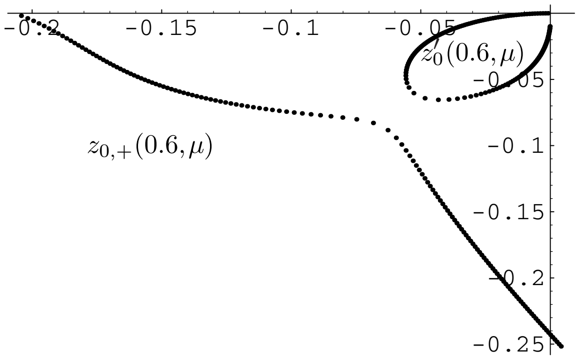


Figure 12. Zeros z_0' and $z_{0,+}$, for $u = 1 - \mathrm{i}$, $\lambda = 0.4 < \lambda_2^*$ and $\mu \in]0, \mu_c(0.4)]$.

If this zero is followed beyond $\mu_c(0.4)$, μ going round this point, then it enters the third sheet of f, that is $\mathbb{R}^+$ is crossed twice clockwise.

Figure 11 also describes a zero of $\hat{f}$. We note that f, on one hand, and f_+, its continuation after $\mathbb{R}^+$ has been crossed once clockwise, on the other, take close values at the same point if μ is small. They are equal if $\mu = 0$. Starting with $z_0(\lambda) = -d(\lambda)$, which is a zero for $\mu = 0$, we get two different zeros depending on whether it is f or f_+ that is considered. z_0 is the zero for f; the zero for f_+ was denoted by $z_{0,+}$ in [13]. It is this latter case that is shown in figure 11.

Up to now we could not find a value λ_2^* of λ for which the two curves of figures 10 and 11 would touch at a point z_2^* for a certain $\mu = \mu_2^*$. Then $(\lambda_2^*, \mu_2^*, z_2^*)$ would have been a solution for (Σ).

In contrast, in the case $u = 1 - \mathrm{i}$, such a point does exist. We have seen numerically that this point exists for $u = x - \mathrm{i}$ when $x > 0.65$. This second solution z_2^* seems to become large when x approaches 0.62; this point could be investigated. In the following section, we shortly present the case $u = 1 - \mathrm{i}$ so as to show how all the zeros of $F(\lambda, \mu, .)$ may be linked.

3.3. Another example of a function g (poles at $1 \pm \mathrm{i}$*)*

Let us give a brief description of the situation in this case where two singular points are present.

The singular point $(\lambda_1^*, \mu_1^*, z_1^*)$ we found in the case $u = -\mathrm{i}$ is displaced; the new values are approximately

$$\lambda_1^* = 0.246 \qquad \mu_1^* = 0.720 \qquad z_1^* = 0.925 - 0.281\mathrm{i}.$$

The second solution for (Σ) is approximately

$$\lambda_2^* = 0.558 \qquad \mu_2^* = 0.085 \qquad z_2^* = -0.070 - 0.089\mathrm{i}.$$

For $\lambda = \lambda_2^*$, it is the two arcs corresponding to the zeros $z_0'(\lambda, .)$ and $z_{0,+}(\lambda, .)$, for $\mu \in]0, \mu_c(\lambda)]$, which touch at z_2^* for $\mu = \mu_2^*$. When λ increases beyond λ_2^*, a qualitative change in the curves occurs, and we change from figure 12 for $\lambda < \lambda_2^*$ to figure 13 for $\lambda > \lambda_2^*$. Here too, (λ_2^*, μ_2^*) is a branch point, but for two determinations of the unique analytic function that $z_0'(., .)$ and $z_{0,+}(., .)$ define.

In the neighbourhood of $\lambda = \lambda_1^*$, the same thing as for $u = \mathrm{i}$ occurs. Finally, according to the position of λ with respect to the two singular points λ_1^* and λ_2^*, we get the three kinds of pictures shown in figure 1 for the zeros of $\hat{f}$.

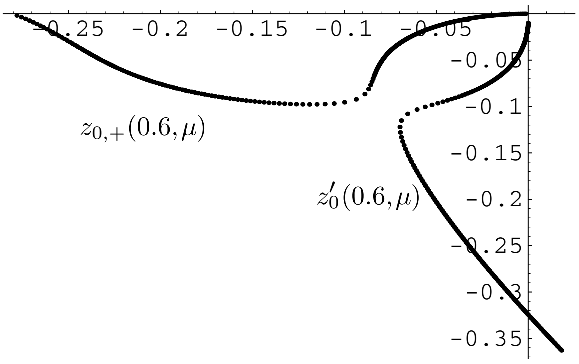


Figure 13. Zeros z_0' and $z_{0,+}$, for $u = 1 - \mathrm{i}$, $\lambda = 0.6 > \lambda_2^*$ and $\mu \in]0, \mu_c(0.6)]$.

When looking at this figure, we can say that, given λ and μ, we have three zeros if $\mu > \mu_c(\lambda)$, and four if $\mu < \mu_c(\lambda)$, for the unique determination $\hat{f}$ of F. The fourth zero is likely to go into the third sheet for $\mu > \mu_c(\lambda)$.

It has been possible to give each zero a name because we considered a particular path in $\mathbb{C}^2$ such that the different kinds of zeros never coincide. But we saw in the ($u = -\mathrm{i}$)-case that the denominations cannot be kept if the coupling constant is to be varied.

For completeness, we now come back to our first example, $u = -\mathrm{i}$ and describe in sections 3.4 and 3.5 some other branch points. We consider the singularities of the zeros obtained when either μ or λ is fixed at an arbitrary real value, different from the preceding 'absolute' singular values μ_i^* and λ_i^*. Even though we do not have any physical interpretation for the complex singular values $\lambda_i^*(\mu)$ and $\mu_i^*(\lambda)$, this may help in understanding the structure of the zeros.

3.4. The case where μ is fixed at a real positive value; branch points in the λ complex plane

Let us come back to the $u = -\mathrm{i}$ case.

3.4.1. A branch point $\lambda_1^*(1)$. Let $\mu = 1$. A calculation on computer shows that (Σ) has at least the solution

$$\lambda_1^*(1) = 0.4830 - 0.0314\mathrm{i} \qquad z_1^*(1) = 0.7397 - 0.4458\mathrm{i}.$$

It is a 2-order branch point for the zeros z_0 and z_1 which interchange when λ goes round that value. Indeed, let $\lambda = \lambda_1^*(1) + 0.01\,\mathrm{e}^{\mathrm{i}\theta}$. When θ varies from 0 to 2π, a machine computation yields that one of the zeros for $\lambda = \lambda_1^*(1) + 0.01$ is $z_0(\lambda_1^*(1) + 0.01, 1) = 0.650 - 0.381\mathrm{i}$ for $\theta = 0$ and changes to $0.838 - 0.509\mathrm{i}$ when $\theta = 2\pi$. In the same way, the latter changes into the former when θ varies from 0 to 2π. The notation of these zeros is of course tricky.

3.4.2. The branch point $\lambda_c^+(1)$. There are also singularities in λ for $\lambda^2 = C^{-1}\mu$, that is, if $\mu = 1$ for $\lambda = \lambda_c^+(1) \simeq \pm 1.253$, since one of the zeros of $F(\lambda, \mu, .)$ comes to 0, the branch point of this function. It is the critical coupling constant of [16]. Let us consider a path around $\lambda_c^+(1)$. Note that $\lambda_c^+(1) > \lambda_1^*$. Let $\lambda = 1.25 - 0.05\,\mathrm{e}^{\mathrm{i}\theta}$ with θ varying from 0 to 2π. The variation of the zeros with μ, for $\lambda = 1.2$ or $\lambda = 1.3$, is a figure 4-type variation for z_0 and z_1 and a figure 11-type one for $z_{0,+}$. For $\lambda = 1.2$ (and $\mu = 1$), the three zeros have the following values:

$$z_{0,+} = -1.48 - 1.32\mathrm{i} \qquad z_0 = 0.0316 - 0.0021\mathrm{i} \qquad z_1 = 2.02 - 0.86\mathrm{i}.$$

The zero we denoted by z_0' does not appear here because it is not in the same sheet.

After one complete turn, as regards z_1, we get back to the initial value; the same for $z_{0,+}$. In contrast, $z_0(1.2, 1)$ changes first into a zero of f when θ reaches a certain value; then it takes a real negative value for $\lambda = 1.3$, goes into the lower half-plane and finally becomes a zero of f_- for $\theta = 2\pi$. (f_- is the continuation of f in $\mathbb{C}\backslash\mathbb{R}^+$, from the lower half-plane, crossing $\mathbb{R}^+$ once anti-clockwise.)

3.4.3. The singularity at $\lambda = 0$. We will not tackle this problem in this paper.

3.5. The case where λ is fixed at a real positive value; branch points in the μ complex plane

3.5.1. A branch point $\mu_1^*(\lambda)$. For $\lambda = 0.4$, for instance, (Σ) has the solution

$$\mu_1^*(0.4) \simeq 0.95 + 0.18\mathrm{i} \qquad z_1^*(0.4) = 0.79 - 0.40\mathrm{i}.$$

Let us determine how the zeros move when μ makes a complete turn around $\mu_1^*(0.4)$, for example, along the path $\mu = \mu_1^*(0.4) + 0.1\,\mathrm{e}^{\mathrm{i}\theta}$. For $\theta = 0$, $z_0(0.4, 1.05 + 0.18\mathrm{i})$ will denote the zero obtained from $z_0(0.4, 0)$ through an analytic continuation which successively follows the real and imaginary directions. We have $z_0(0.4, 1.05 + 0.18\mathrm{i}) \simeq 0.762 - 0.693\mathrm{i}$. In the same way, we define $z_1(0.4, 1.05 + 0.18\mathrm{i})$ from $z_1(0.4, 0)$ through analytic continuation and have $z_1(0.4, 1.05 + 0.18\mathrm{i}) \simeq 0.829 - 0.214$. When θ changes from 0 to 2π, the two zeros $z_0(0.4, 1.05 + 0.18\mathrm{i})$ and $z_1(0.4, 1.05 + 0.18\mathrm{i})$ interchange. Therefore, μ_1^* is a 2-order branch point.

3.5.2. The real branch point $\mu_c(\lambda)$. Let us still assume that $\lambda = 0.4$. If μ makes a complete turn around $\mu_c(0.4) \simeq 0.1018$, starting from 0.10 and following the path $\mu = 0.1020 - 0.0010\,\mathrm{e}^{-\mathrm{i}\theta}$, then the zero $z_0(0.4, 0.1)$ moves continuously towards a value which can be shown to be $z_0'(0.4, 0.1010) \simeq (-5.23 - 0.21\mathrm{i})10^{-4}$. In turn, $z_0'(0.4, 0.1010)$ changes into a zero of f_{++}, under the same change of μ. (f_{++} is the continuation of f in $\mathbb{C}\backslash\mathbb{R}^+$, from the upper half-plane, crossing $\mathbb{R}^+$ twice clockwise.)

3.5.3. The branch point $\mu = 0$. A complete turn anticlockwise around $\mu = 0$ changes $z_0(\lambda, \epsilon)$ into $z_{0,+}(\lambda, \epsilon)$. (See the proof of the lemma.)

4. Physical applications

The principal result of sections 2 and 3 is that there are singular values λ_i^* of the coupling constant, and μ_i^* of the continuum width. Other values $\lambda_c^+(\mu)$ and $\mu_c(\lambda)$ are also of interest. Let us look at the role that all these values play in simple physical models. Only one λ_i^* will be considered in what follows.

4.1. Application to models in electrodynamics

Here, the coupling constant is fixed. In the first model, we will see that it lies below the critical λ^* and can be either above or below $\lambda_c^+(\mu)$. This latter value is itself above or below λ^*, depending on μ.

4.1.1. A model with a harmonic oscillator in three dimensions coupled to the photon. This model was presented in [11]. We will describe it briefly below. Although it is a rough model, it has the interest of yielding a function g which has some physical grounds and on which we can apply the preceding results.

4.1.1.1. The model and its function f. The model consists of a harmonic oscillator in three dimensions coupled to the photon by the interaction $-(e/m)\vec{p}\cdot\vec{A}$, with two simplifications on the Hamiltonian. The first simplification consists in neglecting non-resonant terms (RWA approximation). The second one consists in neglecting excited states of the oscillator except for the first one.

The parameter μ naturally appears in the problem (see [11]); it is $\frac{\hbar}{E\delta}$, where E and δ are, respectively, the energy-level spacing of the oscillator and the distance characterizing the spreading in space of the wavefunctions (exponential part in $\mathrm{e}^{-\frac{1}{2}(\frac{r}{\delta})^2}$). In terms of the mass and spring constant of the oscillator, $\mu = \hbar^{-1/2}m^{3/4}k_{\mathrm{r}}^{-1/4}$. The function f which plays a role in this model is

$$f(z) = z - 1 - \frac{\alpha_{\mathrm{el}}}{3\pi}\int_0^\infty \frac{s\,\mathrm{e}^{-\frac{1}{2}s^2}}{z - \frac{\hbar}{E\delta}s}\,\mathrm{d}s \tag{9}$$

(see [11]), where α_{el} is the fine structure constant. Quantities z, s and f are now dimensionless; the resonance and bound-state energies are obtained by multiplying the zeros of (9) by E. We note that the function $g(s) = \sqrt{\frac{s}{2}}\,\mathrm{e}^{-\frac{1}{4}s^2}$ is neither rational nor bounded at infinity in some sectors of the complex plane.

Variation of μ is thus natural in this problem. It is obtained by varying the characteristics of the oscillator, for example δ, or E (see [9] for a recent study of the spectral properties of related Hamiltonians when E is varied).

Beyond this physical meaning of μ, the result in [11] was the existence of the two zeros of (9) we denoted here by $z_0\big(\sqrt{\frac{\alpha_{\mathrm{el}}}{6\pi}},\mu\big)$ and $z_1\big(\sqrt{\frac{\alpha_{\mathrm{el}}}{6\pi}},\mu\big)$. Again, they were constructed by varying μ, the coupling constant being, of course, fixed. But in order to relate the results to known resonances, the coupling constant must be allowed to vary between 0 and α_{el}. When 'α_{el}' and μ may vary simultaneously, or successively, preceding considerations have shown us that we must be careful. The present study allows us to follow the resonances when μ and $\lambda = \sqrt{\frac{\alpha_{\mathrm{el}}}{6\pi}}$ vary along a nearly arbitrary path, real or complex, and thus to relate z_0 or z_1 to the excited state. We are going to show numerically that $Ez_1\big(\sqrt{\frac{\alpha_{\mathrm{el}}}{6\pi}},\mu\big)$, defined by variation in μ, is indeed the resonance that can be associated with the first excited level of the oscillator, a resonance defined by variation in α. This holds for arbitrary μ. It was not obvious *a priori*.

In order to do that, we have to show that $z_1(\alpha,\mu)$ is a univalued function on $[0,\alpha_{\mathrm{el}}]\times\mathbb{R}^+$. We thus have to determine the singular values α^*, μ^*, z^* and see whether $\alpha_{\mathrm{el}} < \alpha^*$ holds. This is done in the following section.

4.1.1.2. A critical width $\mu_c(\alpha_{\mathrm{el}})$, or distance $\delta_c = \hbar E^{-1}\mu_c^{-1}(\alpha_{\mathrm{el}})$. One has $d := -z_0(\lambda) = \frac{1}{2}(\sqrt{1+4\alpha_{\mathrm{el}}/(3\pi)}-1) \simeq 7.73\times 10^{-4}$. In the following discussion, α does not mean (λ,μ) any more, but α_{el}, in (9), when this constant is allowed to vary. It is real.

We are again confronted with the difficulty of solving (Σ). Let us do it graphically. A possible way to proceed is as follows. We start with the graph in $\mathbb{C}$ of $\mu \to z_0(\alpha,\mu)$, given in [11] for $\alpha = 1/137$ and $\mu \in [1/930, 1/600]$. Figure 14 extends this graph of $z_0(\alpha, .)$ for $\mu \in [0.003, 1]$. The critical value μ_c of μ for which z_0 vanishes is $\frac{\alpha_{\mathrm{el}}}{3\sqrt{2\pi}} \simeq 0.000\,97$ (see [11]). The shape of the curve indicates that we are probably following a curve similar to that of figure 3, corresponding to the path $\alpha'_0, \alpha_1, \ldots$ of figure 5. Now let $\mu = 0.01$ be a test point for which we consider the variation of the zero $\xi_0(\alpha, 0.01)$ with α. We obtain a curve which is similar to that of figure 15. This is the E-type curve in figure 8. This indicates that we may look for an 'absolute' critical value μ^* beyond 0.01. Let us then have μ increasing step by step, and look for a transformation of the shape of this curve into the shape of curve 'E' in figure 9.

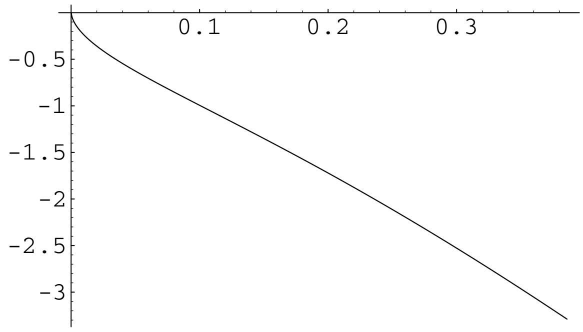


Figure 14. $z_0(\mu)$, $\mu \in [0.003, 1]$, for the 3D oscillator; $\alpha = \alpha_{el} \simeq 1/137$.

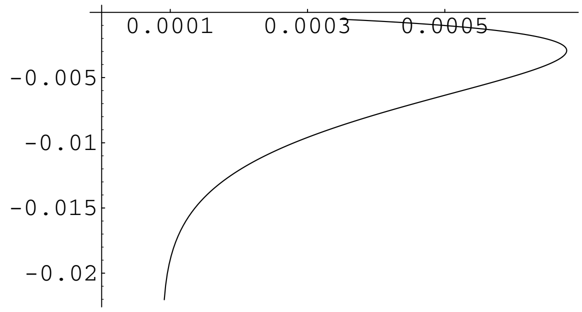


Figure 15. Variation of the resonance position $\xi_0(\alpha, 0.01)$ with the coupling constant $\alpha \in [0.0005, 0.06]$.

The result is given in figures 16(*a*) and (*b*). The change occurs for a value μ^* of μ between 1.036 and 1.037. A first estimate of the corresponding value α^* of α is obtained by following the displacements of the bump in the curve (a part in which the derivative with respect to α is large). It is located between 1.2 and 1.5. We can read that $z^* \simeq 0.87 - 0.64\mathrm{i}$.

A more precise value of α can be obtained by gradually increasing α and determining for which value $z_0(\alpha, .)$ changes from a figure 3-type picture to a figure 4-type picture. We thus get $1.3780 < \alpha^* < 1.3781$.

In summary, if $E = 1$, we get the singular triplet

$$\alpha^* \simeq 1.378 \qquad \mu^* \simeq 1.0368 \qquad z^* \simeq 0.87 - 0.64\mathrm{i}.$$

This μ^* corresponds to an extension in space of the oscillator wavefunctions $\delta^* \simeq 190$ nm. Note that we do not have any physical interpretation of these singular values.

The trajectory of $z_1(\mu)$ is not very easy to find because it is a figure 3-type trajectory and it remains very close to 1, a singular point of the integral (1). But some points showing the shape of the curve are given in figure 17.

Provided that other singular points $\alpha_i^* < \alpha^*$ have not escaped us, this numerical study allows us to relate resonances $z_0(\alpha_{el}, .)$ and $z_1(\alpha_{el}, .)$, built in [11] or here, to excited states built perturbatively in α. We had left this point aside in [11]. The result is as follows:

α_{el} being smaller than α^*, variations with respect to μ and $\alpha \to 0$ commute and, consequently, zeros $z_0(\alpha_{el}, \mu)$ and $z_1(\alpha_{el}, \mu)$ interchange for no value of μ, when α goes to 0. Besides, the former tends to infinity in the lower half-plane, whereas the latter tends to 1. Thus we get the following which had been left unproved in [11]:

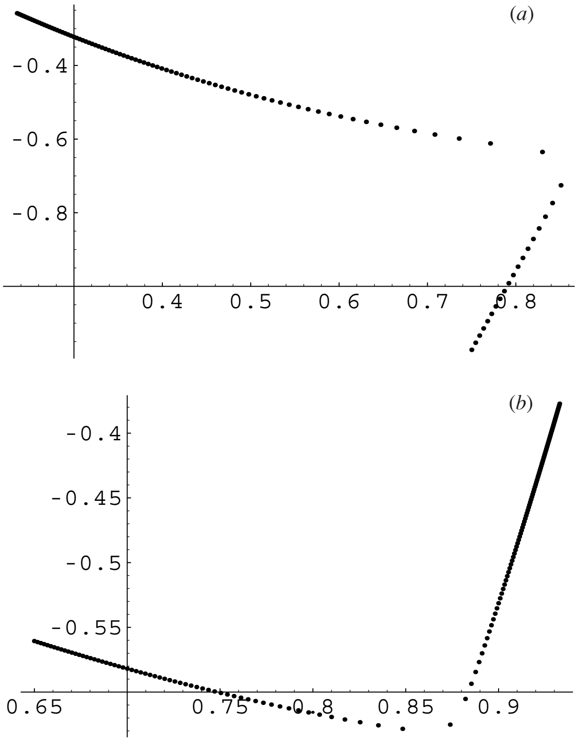


Figure 16. (*a*) $\xi_0(\alpha, 1.036)$, $\alpha \in [1, 3]$ (see section 4.1.1.2); (*b*) $\xi_0(\alpha, 1.037)$, $\alpha \in [1.2, 1.5]$ (see section 4.1.1.2).

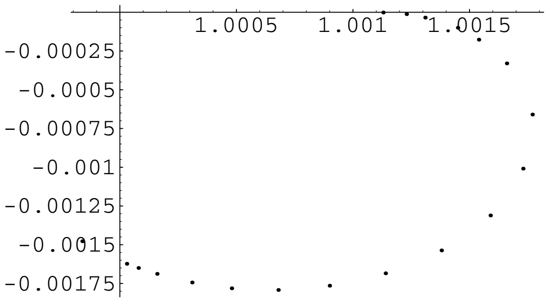


Figure 17. $z_1(\alpha_{\mathrm{el}}, \mu)$ for values of μ in [0.22, 1].

Proposition 5. *$z_1(\alpha_{\mathrm{el}}, .)$ is associated with the first excited level of the decoupled oscillator, for any value of μ. Another resonance exists, $z_0(\alpha_{\mathrm{el}}, .)$, which becomes a bound state when μ becomes smaller than*

$$\mu_c(\alpha_{\mathrm{el}}) = \frac{\alpha_{\mathrm{el}}}{3\sqrt{2\pi}} \simeq 0.000\,97$$

that is when δ becomes greater than

$$\delta_c = \frac{\hbar}{\mu_c(\alpha_{\mathrm{el}})E} \simeq 0.2\,\mathrm{mm}.$$

4.1.2. The hydrogen atom coupled to the photon. In [11], we mentioned that when the hydrogen atom replaces the harmonic oscillator, the simplified model taking only into account the first among the excited levels involves a rational g. Numerical calculations are, therefore, easier and it would be interesting to use electromagnetic matrix elements like (8) to calculate the critical values α_i^* of the coupling constant. However, it is perhaps more important at the moment to improve the theoretical understanding of a multi-level case, so as to be able to get more accurate predictions on δ_c. This could also be useful to get more accurate values of α_i^* and μ_i^* (see section 4.2) in the strong interaction case.

4.2. Behaviour of the resonances when the coupling increases

Let us now see what conclusion may be drawn as regards an arbitrary coupling strength, and thus strong interactions. We are going to answer the question we asked in the introduction about the behaviour of the pole corresponding to the unperturbed excited state as the coupling increases.

Let us consider either the function g with the pole $-\mathrm{i}$, or that of section 4.1.1. In this latter case, the coupling constant is no longer fixed at α_{el}. Physically, the model would still involve a harmonic oscillator, but coupled (by RWA) to a massless boson that would no longer be the photon. In the present state of our analysis, we can describe the situation as follows.

If the coupling constant is small enough, the analysis is similar to that performed with the particular g of section 4.1. In the examples, there is no singular (λ_i^*, μ_i^*) in the rectangle $]0, \lambda_1^*[\times]0, \mu_1^*[$ corresponding to a contact of the z_0 and z_1 curves. Thus, these functions $z_0(\lambda, \mu)$ and $z_1(\lambda, \mu)$ are univalued for $\lambda \in]0, \lambda_1^*[$ and $\mu > 0$ (it is agreed that a detour is needed in the complex μ-plane at $(\lambda, \mu_c(\lambda))$. As a consequence, the notation remains coherent as λ increases up to λ_1^*, and $z_1(\lambda, \mu)$ is indeed associated with the first excited level. In contrast, $z_0(\lambda, \mu)$ is a second resonance.

Let us now have λ increasing to infinity, μ being fixed. We have to distinguish two cases depending on whether μ is smaller or greater than μ_1^*. We will suppose that $\lambda_c(\mu_1^*) > \lambda_1^*$. This was the case for $u = -\mathrm{i}$. We had $\lambda_c(1) \simeq 1.25$, $\lambda_c(1.2) \simeq 1.37$, these two values being greater than λ_1^*. If we had $\lambda_c(\mu_1^*) > \lambda_1^*$, and if (λ_1^*, μ_1^*) corresponded to a multiple point, curves z_0 and z_1 would touch on the real negative axis, which is impossible.

Proposition 6. *If $\mu < \mu_1^*$, whether $\lambda_c(\mu)$ is smaller than, equal to or greater than λ_1^*, the result is qualitatively the same: when λ increases from* 0, *the zero which equals* 1 *when $\lambda = 0$, the complex point usually associated with the excited level, moves into the complex plane and goes to infinity (branch F of figure 8); another zero comes from a neighbourhood of* $-\mathrm{i}$ *(or from infinity in the oscillator case) for λ close to* 0, *gets to* 0 *for $\lambda = \lambda_c(\mu)$ (branch E of figure 8), and goes to $-\infty$ on the negative real axis. It corresponds to a matter–radiation bound state. In both cases, nothing particular happens if λ goes through λ_1^*.*

If $\mu > \mu_1^*$, it is the zero which coincides with $z_1(0, \mu) = 1$ for $\lambda = 0$ (be careful not to call it z_1 nor z_0 without precaution) which moves in the complex lower half-plane before going through 0 *for $\lambda = \lambda_c(\mu)$ (branch E of figure 9) and to $-\infty$ on the negative real axis. The one coming from* $-\mathrm{i}$ *(or from infinity) remains outside the reals (branch F of figure 9).*

To show the results in a picture which displays the μ-variation, we will use two notations.

(1) As before, indices 0 and 1 and the notation z_0 and z_1 will be for zeros constructed from $\mu = 0$.

(2) In order to connect with the usual perturbative view, we will temporarily call a resonance which tends to the unperturbed excited state energy when λ tends to 0 a 'standard resonance' (S). We will call a resonance which does not tend to the unperturbed excited state energy when λ tends to 0 (and is not the fundamental state with no photon) a 'non-standard' (NS) or non-perturbative resonance. For simplicity, we use the term resonance even if it is actually a bound state.

With these notations, and using figures 3, 4, 6 and 7, the main result of our study may be illustrated with the two diagrams in figure 2. λ is fixed either below or above λ^*. The figure shows the variation of the zeros as μ varies from 0 to a large value. When μ goes to infinity, the zero close to 1 tends to 1. S or NS indicates the behaviour when λ goes to 0.

Details about the variation with respect to λ are given in figures 8 and 9, for two kinds of values of μ. The behaviour when λ goes to infinity is not explicit in figure 2.

5. Conclusion

In a simple model with only one 'excited' level, where the interaction with the quantum electromagnetic field is involved, we saw that at least two resonances/bound states do exist. One is familiar to us and has been called standard. There is at least another one, which we called non-standard, and which is elsewhere in the complex plane when λ tends to 0, the other parameter being fixed. In some examples, where g has only one pole in the lower half-plane, the resonance tends to that pole. If g is $e^{-p^2/2}$, then it goes to infinity.

In this latter case, taken from quantum electrodynamics, if the space extension of the oscillator states increases beyond 0.2 mm, the present study implies that it is the non-standard resonance which becomes a bound state.

We get results beyond the electromagnetic interaction case considering the same type of matter–radiation interaction but with a coupling constant which may be arbitrary. The behaviour of the standard resonance as the coupling constant varies depends on the width of the coupling function. There is a critical width. If we choose a particular shape by choosing a particular g and consider the whole family of μ-dilated functions, we have seen that there are singular real couples (λ^*, μ^*). In three examples (sections 3.2, 3.3, 4.1.1 and 4.2), depending on whether μ is greater than μ^* or not, we have seen two different types of behaviour of the standard resonance when the coupling constant increases. If μ is greater than μ^*, the standard resonance changes into a bound state; if μ is smaller, it does not. Small values of μ probably correspond to large extensions in space of the system coupled to the radiation. These results should be worth testing in actual circumstances.

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